THE ELEVENTH REGULAR MEETING OF THE SOUTHWESTERN SECTION.

THE eleventh regular meeting of the Southwestern Section of the Society was held on December 1, at Norman, Oklahoma, in the administration building of the University of Oklahoma, Professor S. W. Reaves presiding. The sessions opened at 10.30 A.M. and 2.30 P.M. respectively. Attending members were entertained at a smoker on the previous evening and at lunch on the day of the meeting. There were present fourteen persons, including the following eleven members of the Society:

Professor Nathan Altshiller, Professor C. H. Ashton, Professor Henry Blumberg, Professor E. W. Davis, Professor E. P. R. Duval, Professor E. R. Hedrick, Professor O. D. Kellogg, Mr. B. B. Libby, Mr. E. D. Meacham, Professor S. W. Reaves, and Professor W. H. Roever.

It was decided that the next meeting of the Section should be held on November 30, 1918, at Columbia, Missouri, and that the program committee should be Professors E. R. Hedrick (chairman), W. C. Brenke, O. D. Kellogg (secretary).

The following papers were presented at this meeting:

(1) Professor A. M. HARDING: "Rational plane anharmonic cubics."

(2) Professor NATHAN ALTSHILLER: "On the I-centers of a triangle."

(3) Professor R. D. CARMICHAEL: "Fermat numbers $F_n = 2^{2^n} + 1$."

(4) Dr. P. R. RIDER: "An intrinsic equation solution of a problem of Euler."

(5) Professor O. D. KELLOGG: "Interpolation properties of solutions of certain differential equations."

(6) Professor S. LEFSCHETZ: "On multiple integrals belonging to an algebraic variety."

(7) Professor NATHAN ALTSHILLER: "On the Teixeira construction of the unicursal cubic."

(8) Professor HENRY BLUMBERG: "Non-measurable functions connected with functional equations."

(9) Professor HENRY BLUMBERG: "A theorem on semi-continuous functions."

(10) Professor J. E. MCATEE: "Polynomial modular invariants of a binary quadratic."

(11) Professors E. R. HEDRICK and LOUIS INGOLD: "A generalization of Bessel's inequality and related formulas."

(12) Professor W. H. ROEVER: "Geometric explanation of a certain optical phenomenon."

Professor McAtee's paper was communicated to the Society through Professor Fleet. In the absence of the authors the papers of Professors Harding, Carmichael, Lefschetz, and McAtee were read by title, and Dr. Rider's paper was presented by Professor Roever.

Abstracts of the papers follow in the order of their titles above.

1. In a recent paper (*Giornale de Matematiche*, volume 54, 1916) Professor Harding has studied certain projective properties of anharmonic curves and has found the coordinates of the invariant triangle in terms of the invariants of a certain differential equation. The present paper is limited to a discussion of the properties of rational anharmonic cubic curves. It is shown that each of these curves has one cusp and one point of inflection, and that the invariant triangle is formed by the cuspidal tangent, the inflectional tangent, and the line joining the cusp to the point of inflection.

2. Professor Altshiller's paper will soon appear in the *American Mathematical Monthly*.

3. In this paper, Professor Carmichael gives elementary proofs of all the essential known facts about the Fermat numbers $F_n = 2^{2^n} + 1$ and a derivation of some new results of minor importance.

4. The following problem was proposed and solved by Euler: Given two points P_0 and P_1, and directed lines P_0Q and QP_1 through them, to determine an arc tangent to these two lines at P_0 and P_1, which with its evolute and its normals at P_0 and P_1 will enclose the minimum area. Making use of a method developed by Radon for minimizing the integral $\int_{s_0}^{s_1} F(x, y, \theta, \kappa)ds$, Dr. Rider gives an intrinsic equation solution of the problem which is much shorter than the solution ordinarily given.

5. Continuing the work reported on at the summer meeting, (see BULLETIN, November, 1917, page 62) Professor Kellogg

extends to the case of the general self-adjoint homogeneous boundary conditions the results of Liouville on the possession of the interpolation property (D) by the solutions of linear differential equations of second order. The results will appear in the *American Journal of Mathematics*.

6. Let $F(x, y, z, t) = 0$ be the equation of an algebraic three-dimensional variety in S_4 and consider the integrals

$$J = \int\int\int \frac{P(x, y, z, t)}{F_t'}\, dxdydz, \quad J = \int\int \frac{P(x, y, z, t)}{F_t'}\, dxdy$$

belonging to F and to its hyperplane sections $z = C$, or H_z. The fundamental theorem proved by Professor Lefschetz in his paper is this: The periods of J_z are algebraic functions of z. This makes it possible to replace J as far as its periods are concerned by a certain abelian integral. The number of these periods is found to be $R_3' = I + 2R_2 - 3R_1 - 4$ (I invariant of Zeuthen-Segre, R_i i-dimensional connectivity). By comparing with a result of Alexander it is seen that $R_3' = R_3 - R_1$, which agrees with the existence of R_1 tridimensional cycles relative to which the periods of J are zero. The definition of triple integrals of the second kind is about the same as Picard's for algebraic surfaces. Their reduction and enumeration is closely related to a class of double integrals of the type

$$\int\int U dydz + V dzdy + W dxdy; \qquad \frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} + \frac{\partial W}{\partial z} = 0,$$

and to an invariant number λ analogous to Picard's number ρ. The number of integrals of the second kind turns out to be $\rho_0 = R_3' - \lambda$. Extensions to higher varieties are also considered. A summary of these results appeared in the *Comptes Rendus* of May 29, 1917.

7. Consider a point O, a line s, a conic C, and two points D and A on s and C respectively. A variable line through A meets s in P and C again in B. Let $M \equiv (OP, DB)$. The locus of the point M is, in general, a unicursal cubic having O for its double point, passing through D and through the points common to s and C. Conversely, with an arbitrarily chosen straight line an infinite number of conics may be associated in

order to generate a given unicursal cubic by the above method, provided the straight line does not pass through the double point of the cubic and is not an inflectional tangent. Professor Altshiller proves synthetically and discusses this proposition, which is a generalization of a theorem due to F. Gomes Teixeira (*Nouvelles Annales de Mathématiques*, August, 1917, pages 281–284).

8. At the 1916 meeting of the Southwestern Section, Professor Blumberg communicated—among other things—the fact that discontinuous solutions of the functional equation $f(x+y) = f(x) + f(y)$ are necessarily non-measurable. This holds essentially also for the equation $f(xy) = f(x)f(y)$. These examples suggest the generalized relation

$$f[\varphi(x, y)] = \varphi[f(x), f(y)],$$

where φ is to be regarded as a fixed function of two variables and f as the variable solution of the functional equation. Under suitable assumptions for φ, a large class of functional equations is obtained whose discontinuous solutions are necessarily non-measurable.

9. The theorem of Professor Blumberg's second paper is as follows: Let φ_{ab} be a monotone decreasing, real interval-function; i. e., φ_{ab} is a real number for every (closed) interval (a, b), and, in addition, $\varphi_{ab} \leqq \varphi_{cd}$ for (a, b) within (c, d). Moreover, let φ_{ai} have a finite lower bound. Let $\varphi(x)$ be the point function associated with φ_{ab}; i. e., $\varphi(x)$ is the greatest lower bound of all φ_{ab} such that x is in the interior of (a, b); let $\varphi^{(r)}(x)$ —the symbol (r) designating "right"—be the greatest lower bound of all φ_{xb}, and $\varphi^{(l)}(x)$, the greatest lower bound of all φ_{ax}. Then each of these three functions of x is an upper semicontinuous function—this part of the theorem is in essence not new—and $\varphi(x) = \varphi^{(r)}(x) = \varphi^{(l)}(x)$ except at most in a countable set. It follows that the saltus function $s(x)$ is, except at most in a countable set, indentical with $s^{(r)}(x)$ and $s^{(l)}(x)$—whose meaning is self-explanatory—and similarly for the f-saltus, d-saltus, etc. (Cf. author's paper, "Certain general properties of functions," *Annals of Mathematics*, March, 1917.) As a very special case, it follows that a function continuous on the right (left) except at most at the points of a countable set is everywhere continuous except possibly at the points of a countable set.

10. Professor Dickson has found certain polynomial modular invariants and properties of such invariants for the case of a binary n-ic and a prime modulus P. In this paper Professor McAtee generalizes these invariants for the case of a binary quadratic and modulus a power of a prime, P^λ. Then these invariants are specialized for the case $P = \lambda = 2$ and a fundamental system is exhibited modulo 4.

11. With a view toward applications to the expansion of functions in terms of given functions, Professors Hedrick and Ingold have been led to study the properties of a general linear distributive operation $L[f(x), \varphi(x)]$ on a pair of functions $f(x)$ and $\varphi(x)$. In this paper, it is shown that such an operation leads at once to a general formula of which Bessel's inequality is a special case. Other related formulas are also generalized.

A similar operator, defined only for the product $f(x) \cdot \varphi(x)$ has been studied by Moore. (See BULLETIN, volume 18, pages 334–362.)

12. Professor Roever gives a geometric explanation of elliptical light curves seen in a highly scratched plate illuminated by a point source.

O. D. KELLOGG,
Secretary of the Section.

NOTE ON CONJUGATE NETS WITH EQUAL POINT INVARIANTS.

BY DR. G. M. GREEN.

(Read before the American Mathematical Society, September 4, 1917.)

IN my second memoir on conjugate nets on a curved surface,* I gave a new characterization of conjugate nets with equal Laplace-Darboux invariants. The theorem as there stated, however, is not quite complete, and it is the purpose of this note to supply the necessary refinement, as well as to generalize the theorem and put it into relation with another,

* *Amer. Journal of Mathematics*, vol. 38 (1916), pp. 287–324. See in particular p. 313.

which is involved in a different characterization of conjugate nets with equal invariants.

Using the notation of the paper just cited, let

$$y^{(k)} = y^{(k)}(u, v) \qquad (k = 1, 2, 3, 4)$$

be the point equations, in homogeneous coordinates, of a surface S referred to a conjugate net. The four functions y are a fundamental system of solutions of the completely integrable system of partial differential equations

$$\begin{aligned} y_{uu} &= ay_{vv} + by_u + cy_v + dy, \\ y_{uv} &= b'y_u + c'y_v + d'y. \end{aligned} \tag{1}$$

The minus first and first Laplace transforms of the point y of the surface are respectively

$$\rho = y_u - b'y, \quad \sigma = y_v - a'y.$$

The line joining these two points we shall call with Wilczynski the ray of the point y, and the congruence of rays the ray congruence. The developables of the ray congruence correspond to a net of curves on S, which we shall call the ray curves, and whose differential equation is*

$$aHdu^2 - \mathfrak{D}dudv - Kdv^2 = 0, \tag{2}$$

where

$$H = d' + b'c' - b_u', \quad K = d' + b'c' - c_v'$$

are the Laplace-Darboux invariants, and

$$\mathfrak{D} = d + ab'^2 - c'^2 + ab_v' - c_u' + b'c + bc'.$$

Two ray curves pass through each point y of the surface, and we shall call their tangents at the point y the ray tangents of y. The ray tangents are the lines which join y to the points $y_u + (dv/du)_1 y_v$, $y_u + (dv/du)_2 y_v$, where $(dv/du)_1$ and $(dv/du)_2$ are the roots of equation (2) considered as a quadratic in dv/du.

The focal points of the ray $\rho\sigma$ are given by the formulas†

$$R = \rho + r_1\sigma, \quad S = \rho + r_2\sigma,$$

where r_1 and r_2 are the roots of the quadratic

$$Hr^2 + \mathfrak{D}r - aK = 0. \tag{3}$$

* G. M. Green, loc. cit., p. 309.

† Loc. cit., p. 309.

A comparison of equations (2) and (3) will show that *the ray tangents meet the ray in the focal points of the ray if and only if the conjugate net has equal Laplace-Darboux invariants, provided, however, that* $\mathfrak{D}$ *is not zero.** If $\mathfrak{D}$ is zero, the quadratics (2) and (3) would coincide not only if $H = K$, but also if $H = -K$. If, however, $H = -K$, and $\mathfrak{D}$ is zero, equation (2) becomes

$$H(adu^2 + dv^2) = 0.$$

If, then, H is different from zero (if it were zero, we should have again $H = K$) the differential equation of the ray curves becomes

$$adu^2 + dv^2 = 0,$$

which is the differential equation of the asymptotics. In other words, *if the ray curves coincide with the asymptotic curves, then the ray tangents will meet the ray in the focal points of the ray.*

However, the geometric facts enable us to state the entire situation more sharply. An analytic proof of these facts—we shall presently supply a geometric proof—will show that in the first case, i. e., $H = K$, the tangent to one of the ray curves through y meets the ray in the focal point which corresponds to the other ray curve, whereas in the second case, i. e., $\mathfrak{D} = 0$ and $H = -K$, the tangent to a ray curve meets the ray in the focal point belonging to that ray curve. So we may state our characterization of conjugate nets with equal invariants as follows:

A necessary and sufficient condition that a conjugate net have equal point invariants is that for every point of the surface the tangents to the two ray curves at the point meet the corresponding ray in the focal points of the ray, the tangent to either ray curve meeting the ray in the focal point which corresponds to the other ray curve. If each ray tangent meets the ray in the focal point which corresponds to it, the ray curves coincide with the asymptotics of the surface. In this case $H = -K$ *and* $\mathfrak{D} = 0$. *If* $H = K = \mathfrak{D} = 0$, *the ray curves are indeterminate.*

In this connection it will be well to recall Wilczynski's theorem,† that *a necessary and sufficient condition that a con-*

* Loc. cit., p. 313, except for the proviso at the end of the present statement.

† E. J. Wilczynski, "The general theory of congruences," *Transactions Amer. Math. Society*, vol. 16 (1915), pp. 311–327.

jugate net have equal point invariants is that its ray curves form a conjugate net. Using this theorem, we may prove the previous one on the basis of the following very general theorem:

Let t_1 and t_2 be the ray tangents of a point of the surface, and Φ_1, Φ_2 the corresponding focal points, respectively, of the ray. Then the ray is met in Φ_1 by the tangent to the surface which is conjugate to t_1, and in Φ_2 by the tangent conjugate to t_2.

We shall give a geometric proof of this theorem; the proof will show that a perfectly good theorem may sometimes be obtained by merely dualizing a trivial statement. Let us regard the surface S as the locus of points y, and the envelope of the corresponding tangent planes Π. By polar reciprocation, S goes into a surface S_1, the point y becoming the tangent plane Π_1, and the plane Π becoming the point of contact y_1 on S_1. The points ρ and σ become the osculating planes of the parametric curves $u = \text{const.}$, $v = \text{const.}$, respectively, at the point y_1 on S_1. These osculating planes meet in a line, which Wilczynski has called the axis of the point y_1. The totality of axes constitute the axis congruence, and the curves on S_1 which correspond to the developables of this congruence are called the axis curves. The points of a curve C on S, i. e., a one-parameter family of points, correspond to a one-parameter family of planes tangent to S_1. The points of contact of these planes constitute a curve C_1 on S_1, which we may consider as the curve corresponding to C. But the tangent to C at y does not go over into the tangent to C_1 at y_1, but into the conjugate direction. The focal points of the ray $\rho\sigma$ correspond to the focal planes* of the axis congruence of S_1. An axis tangent on S_1, i. e., a tangent to an axis curve, corresponds to the direction on S which is conjugate to a ray tangent on S. If we recall that a focal plane of a line of a congruence is tangent to a developable of the congruence all along a line of the congruence, and if we observe that the developables of the axis congruence actually cut the surface S_1 in the axis curves, then it becomes evident that *the axis tangents lie in the corresponding focal planes.* If we dualize this situation on S_1, we obtain the situation on S which is described in the theorem which we set out to prove.

Having, then, shown that the direction conjugate to one ray tangent meets the ray in that focal point of the ray which

* The focal planes of an axis are the planes tangent, along the axis, to the two developables of the axis congruence to which that axis belongs.

corresponds to the other ray tangent, and recalling Wilczynski's theorem, we are led at once to the final form in which we stated the geometric characterization of conjugate nets with equal point invariants. This characterization may of course be established analytically, and independently of Wilczynski's theorem, so that, on the basis of our general theorem, Wilczynski's characterization may then be obtained from ours.

HARVARD UNIVERSITY,
August 22, 1917.

ON THE DIFFERENTIABILITY OF ASYMPTOTIC SERIES.

BY DR. J. F. RITT.

(Read before the American Mathematical Society, October 27, 1917.)

THE question of the differentiability of asymptotic series seems not to have received adequate treatment.* Writers on the theory of asymptotic convergence content themselves always with stating that if $P(x)$ has the asymptotic representation

$$P(x) \sim a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + \cdots,$$

it may not have a derivative at all, and that even if a derivative does exist, the derivative may not admit of asymptotic development.

A failure to distinguish between the real and complex domains, in this connection, is responsible for a serious lacuna, which it is the purpose of this note to fill.

Let $P(x)$, defined in a sector with vertex at the origin, be analytic within the sector, in the neighborhood of the origin, and continuous on the sides of the sector, and at the origin. Let $P(x)$ have the asymptotic development of finite order

$$P(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + \epsilon(x) x^n,$$

where $n \geqq 2$ and where $\epsilon(x)$ goes to zero with x. We say that

* Since writing this note, I have ascertained by correspondence with Professor Birkhoff that he was familiar with and had used the result given here, but failed to publish it, being under the impression that it was contained in an article by W. B. Ford (*Bull. de la Soc. Math. de France*, 1911, p. 347).

$P'(x)$, the derivative of $P(x)$, which, of course, exists in the interior of the sector, exists also at the origin relative to any sector within the given one, and that in the inner sector $P'(x)$ has the development

$$P'(x) = a_1 + 2a_2x + 3a_3x^2 + \cdots + (n-1)a_{n-1}x^{n-2} + \epsilon_1(x)x^{n-2}.$$

The proof is simple. Put $x = 1/y$. Then $P(1/y)$ is defined and is analytic, for sufficiently large values of y, in a sector S, which is very simply related to the sector of definition of $P(x)$, and has the asymptotic development

$$P(1/y) = a_0 + \frac{a_1}{y} + \frac{a_2}{y^2} + \cdots + \frac{a_n}{y^n} + \frac{\epsilon(1/y)}{y^n}.$$

We have

$$\frac{d}{dy}P(1/y) = -\frac{a_1}{y^2} - \frac{2a_2}{y^3} - \cdots - \frac{na_n}{y^{n+1}} - \frac{n\epsilon(1/y)}{y^{n+1}} + \frac{1}{y^n}\frac{d}{dy}\epsilon\left(\frac{1}{y}\right).$$

Consider now any sector S' interior to S, the sector of definition of $P(1/y)$. For sufficiently large values of y any point of S' is at least at a distance of unity from the sides of S. For any such point in S', we have

$$\frac{d}{dy}\epsilon(1/y) = \frac{1}{2\pi i}\int_c \frac{\epsilon(1/\zeta)d\zeta}{(\zeta - y)^2},$$

the integration being made around a circle of radius unity, so that

$$\left|\frac{d}{dy}\epsilon(1/y)\right| < M,$$

where M is the maximum modulus of $\epsilon(1/y)$ along the circle. Since $\epsilon(1/y)$ goes to zero as y becomes infinite, it is clear that we may collect the last three terms in the expression above for the derivative of $P(1/y)$, and write

$$\frac{d}{dy}P(1/y) = -\frac{a_1}{y^2} - \frac{2a_2}{y^3} - \cdots - \frac{(n-1)a_{n-1}}{y^n} - \frac{\epsilon_1(1/y)}{y^n},$$

where $\epsilon_1(1/y)$ goes to zero with $1/y$.

Reverting now to $P(x)$,

$$P'(x) = \frac{d}{dy}P(1/y)\frac{dy}{dx} = -y^2\frac{d}{dy}P(1/y)$$

$$= a_1 + \frac{2a_2}{y} + \cdots + \frac{(n-1)a_{n-1}}{y^{n-2}} + \frac{\epsilon_1(1/y)}{y^{n-2}},$$

and finally

$$P'(x) = a_1 + 2a_2x + \cdots + (n-1)a_{n-1}x^{n-2} + \epsilon_1(x)x^{n-2},$$

which is the asymptotic development sought. Also it results easily from the fact that $P'(x)$ approaches a_1 as x goes to zero that the derivative at the origin exists and is equal to a_1.

It is interesting to notice that a single differentiation has lost us two terms of the development of $P(x)$.* However, if $P(x)$ has an infinite asymptotic development, it is clear that $P'(x)$ will also have an infinite development, and in fact that the development of $P(x)$ can be differentiated formally any number of times.

The above proof applies, of course, to other domains than sectors; for instance, we might use the horn angle obtained by an inversion relative to the origin, and a reflection across the real axis of the infinite strip between any two parallel lines.

Lastly, it is important to notice that we have also established above the differentiability in the complex domain of asymptotic developments in descending powers of the variable.

COLUMBIA UNIVERSITY.

DARBOUX'S CONTRIBUTION TO GEOMETRY.

BY PROFESSOR L. P. EISENHART.

(Read before the joint session of the American Mathematical Society and the Mathematical Association of America at Cleveland, Ohio, September 6, 1917.)

GASTON DARBOUX was born in 1842 at Nîmes, a place of interest to mathematicians because here from 1819 to 1831 Gergonne edited his *Annales*, and incidentally exerted a great influence on the development of geometry. At the age of eighteen Darboux went to Paris, in whose intellectual life he had a prominent part for fifty-seven years. As a student, first at the Ecole Polytechnique and then at the Ecole Normale, his unusual mathematical ability made him conspicuous. His

* The loss of the term $na_n x^{n-1}$ is only apparent. This can be shown by taking the radius of the circle of integration above equal to $k|y|$, where k is some number independent of y. I prefer the proof above because of its applicability to more general domains than sectors.

rare powers of expression soon won for him a reputation as a lecturer and expositor, with the consequence that he received desirable teaching posts from the beginning. In 1880 he succeeded Chasles in the chair of Géométrie supérieure at the Sorbonne. Four years later he became a Membre de l'Institut, and in 1889 assumed the duties of Doyen de la Faculté des Sciences. At the death of Bertrand in 1900 Darboux was elected Secrétaire Perpétuel de l'Académie des Sciences, which post he administered with distinction until his death. We cannot go further into detail concerning the many honors and responsibilities which came to him in his very full life. For we are concerned with the work he did in the field of geometry, to a review of which we now turn.

While a student at the Ecole Normale in 1864, Darboux published his first papers, two notes in the *Nouvelles Annales*.* It is important that we consider these papers in some detail, as they contain germs of his subsequent work. The first dealt with the plane sections of the tore. It was shown that the curve is of the fourth order with the circular points at infinity for double points; that there are sixteen foci, in the sense of Plücker, of which four are real and lie on a circle; that there is a homogeneous linear relation between the distances of any point of the curve from three of these foci; and that an inversion with respect to any of these foci as pole transforms the curve into an oval of Descartes.

In his second note Darboux considered the curves of intersection of a sphere and a quadric. Through such a curve pass four quadric cones; the tangent planes of the cone meet the sphere in circles doubly tangent to the curve; the four of these planes tangent also to the sphere determine four null circles doubly tangent to the curve and lying on a circle; the distances of any three of them from any point of the curve are in homogeneous linear relation. The curve has, therefore, sixteen foci, lying in sets of four on four circles, such that any two circles meet orthogonally. To these curves and their transforms by inversion Darboux has given the name cyclics; they are spherical or plane and are of the fourth order. In particular, the plane sections of the tore and the ovals of Descartes are cyclics, as are also the cissoid, lemniscate, circular cubic, and other well-known curves.

In 1872 Darboux published his first treatise, entitled Sur une

* Ser. 2, vol. 3 (1864), pp. 156, 199.

Classe remarquable de Courbes et de Surfaces algébriques et sur la Théorie des Imaginaires. It consists of five parts, one of which is devoted to a discussion of cyclics.

When Darboux was at the Ecole Normale, he became familiar with the writings of Lamé, Dupin, and Bonnet on triply orthogonal systems of surfaces, the field with which his name will ever be associated. At that time the best example of an orthogonal system of surfaces was afforded by the confocal quadrics. Years before Kummer had sought families of plane curves defined by an equation $f(x, y, a) = 0$, where a is the parameter of the family, such that through each point of the plane pass two curves of the family and that they meet orthogonally. He found that such families have the property that all the curves are confocal. Darboux, seeking to generalize this result, sought the triply orthogonal systems of surfaces, each of which can be defined by a single equation involving a parameter λ. In particular he sought the generalization of the orthogonal family of ovals of Descartes having three common foci. As a result he found the system defined by the single equation

$$(x^2+y^2+z^2)^2+\frac{\alpha\lambda-4h}{\alpha-\lambda}x^2+\frac{\beta\lambda-4h}{\beta-\lambda}y^2 + \frac{\gamma\lambda-4h}{\gamma-\lambda}z^2-h=0, \tag{1}$$

where α, β, γ and h are constants, and λ is the parameter of the system. Each surface of the system so defined, subsequently called cyclides by Darboux, has the following properties: it is of the fourth order and has the imaginary circle at infinity for a double curve; it is cut by any sphere in a cyclic; in five different ways it is the envelope of a two-parameter family of spheres with their centers on a fixed quadric and orthogonal to a fixed sphere, each of the doubly tangent spheres meeting the surface in two circles. These results were presented to the French Academy of Sciences on August 1, 1864.* On the same day Moutard announced to the Academy the discovery of the same system. He had just been studying surfaces transformable into themselves by inversion, and had found that the surfaces of the fourth order with the circle at

* *Comptes Rendus*, vol. 59 (1864), p. 240. The details of this paper were published in *Annales de l'Ecole Normale*, vol. 2 (1865), pp. 55–69.

infinity for double curve are transformable in five different ways. While seeking the lines of curvature of these surfaces he found the triple system of cyclides.

After finding this orthogonal system, Darboux determined the linear element of space in terms of parameters referring to such a system of surfaces, and found, as in the case of confocal quadrics, that the curves of intersection form an isothermic system of curves on each surface. The celebrated theorem of Dupin, that in a triply orthogonal system any two surfaces of different families cut in a line of curvature for both, was made complete by the following addition due to Darboux:

When two families of orthogonal surfaces cut along the lines of curvature of these surfaces, there is a third family orthogonal to the first two families.

With the aid of this theorem Darboux was enabled to determine the condition to be satisfied in order that the family of surfaces defined by an equation of the form $\varphi(x, y, z) = \alpha$ can be part of a triple system; he found that φ must be any solution of a partial differential equation of the third order in two independent variables, but he did not calculate the equation, because of its complicated form. These results were published by Darboux in 1866 in his classic memoir "Sur les surfaces orthogonales"* subsequently presented as his thesis for the doctorate. This memoir contained also the determination of the orthogonal systems for which the lines of curvature are plane, and the erroneous theorem that the triple system of cyclides (1), which includes the system of confocal quadrics as a special case, affords the only example of a triple system of isothermic surfaces, a mistake corrected by Darboux in a later paper.

In 1872 Cayley attacked the problem of orthogonal systems, and gave the differential equation of triple systems the form now associated with his name, and valuable because of its adaptability to the determination of particular orthogonal systems. Darboux was quick to see the value of Cayley's work. On the one hand he made a study of the analytical processes underlying Cayley's equation and extended them to systems in n variables. On the other hand he made use of this equation for the determination of orthogonal systems for which the surfaces in one family are quadrics; those for which

* *Annales de l'Ecole Normale*, vol. 3 (1866), pp. 97–141.

the surfaces in one family have a plane of symmetry; and orthogonal systems containing a given surface and involving in their equation four arbitrary functions of a single variable. These results were published in the first and second parts of his second great memoir on orthogonal systems.* In the third part he generalized the equations of Lamé to space of n dimensions and studied certain special problems in this connection. The last fifty pages of this memoir are devoted to the problem which he thought he had solved in his thesis, namely the determination of orthogonal systems of isothermic surfaces. He found other systems than those of cyclides, but we cannot go further into detail. As it is, we have devoted a great deal of space to these two memoirs, which, however, seems justified in view of their importance in the development of the theory of orthogonal systems, and of their place among Darboux's memoirs. Twenty years later he published his Leçons sur les Systèmes orthogonaux et les Coordonnées curvilignes, which includes much of the foregoing theory.† In the second memoir there appears also a reference to triply conjugate systems of surfaces, studied subsequently by Guichard and Tzitzeica and discussed at length by Darboux in the second edition of the above treatise issued in 1910.‡ In this volume and in subsequent papers Darboux made further contributions to the theory of orthogonal systems.

From the time when Gauss introduced the idea of curvilinear coordinates of a surface and discovered the absolute invariant of the linear element which measures the curvature of the surface, geometers have been interested in the problem, as yet unsolved, of finding all surfaces with a given linear element, that is, the problem of applicable surfaces. In 1872 Darboux§ showed that the rectangular point coordinates of a surface with a given linear element are solutions of a partial differential equation of the second order of the Ampère type; also that when one solution of this equation is known two others can be found by quadratures, and that these three solutions are the coordinates of a surface with the given linear element.

* *Annales de l'Ecole Normale*, ser. 2, vol. 7 (1878), pp. 97–151, 227–261, 275–349.

† A full analysis of this treatise, by E. O. Lovett, was published in this BULLETIN, vol. 5 (1899), pp. 185–202.

‡ A review by Wilczynski of this edition is published in the BULLETIN, vol. 20 (1914), pp. 247–253.

§ Sur une Classe remarquable, etc., pp. 14, 181.

Moutard and Ribaucour, who were at work on the problem of applicable surfaces in the late sixties, showed that when two applicable surfaces are known one can find at once two other surfaces S and S_1 whose rectangular coordinates x, y, z and x_1, y_1, z_1 satisfy the condition

$$dxdx_1 + dydy_1 + dzdz_1 = 0. \tag{2}$$

In 1873 Darboux announced to the Société mathématique de France* that the problem of the infinitesimal deformation of a surface S is equivalent to the determination of the surfaces S_1 whose coordinates satisfy equation (2). In his well-known treatise Leçons sur la Théorie générale des Surfaces† he showed that when the equation (2) for a surface S has been integrated, the infinitesimal deformation of S, to terms of any order, is a problem of quadratures alone. Weingarten‡ reduced the solution of equation (2) to the integration of a linear partial differential equation of the second order, whose equation of characteristics defines the asymptotic lines of the given surface. When the latter are parametric, the equation is of the form

$$\partial^2\theta/\partial u\partial v = \mu(u, v)\theta. \tag{3}$$

A surface S_0 corresponding to a given surface in such a way that the tangent planes to the two surfaces at corresponding points are parallel and asymptotic lines on each surface correspond to a conjugate system on the other is said to be associate to it. This idea is due to Bianchi,§ who proved that the determination of the surfaces associate to a given surface is equivalent to the integration of equation (2). Having noted the reciprocal character of the relations between S and S_1 and between S and S_0, Darboux found nine other surfaces arising from each solution of equation (2), the whole group of twelve surfaces forming a closed system.

It would be a great oversight were we not to refer to the charming chapters in the fourth volume of the Leçons in which Darboux treats the rolling of one applicable surface upon another, and the geometrical configurations thus generated by points, lines, etc., invariably fixed with respect to the rolling surface. Ribaucour also had a part in the develop-

* This paper was not published.
† Vol. 4, p. 5. Hereafter this treatise will be referred to as the Leçons.
‡ *Crelle*, vol. 100 (1886), pp. 296–310.
§ Lezioni di Geometria differenziale, Pisa, 1894, p. 279.

ment of this field and recently Bianchi. As an example of the beauty of some of the theorems in this theory we cite the following: Let S and S_1 be two applicable surfaces; as S rolls over S_1, a null sphere with center invariably fixed with respect to S cuts the common tangent plane to the two surfaces in a circle; these circles form a cyclic system.

The idea of the spherical representation of a surface is due to Gauss. The lines of curvature of a surface are represented on the sphere by an orthogonal system of curves. Darboux early interested himself in the solution of the problem of finding the surfaces whose lines of curvature are represented on the sphere by a given orthogonal system, which he called the "problem of spherical representation." By making use of the simple expressions for the rectangular coordinates of a point of the sphere in terms of parameters referring to the imaginary generators, he showed that the problem is reducible to the solution of a partial differential equation of the form (3). He wrote a number of papers dealing with special forms of this equation, and with the geometrical significance of this work and of the researches of Moutard on the equation (3).

We recall that Moutard showed that a cyclide of the fourth order is, in five different ways, the envelope of a two parameter family of spheres orthogonal to a fixed sphere S, the centers of the spheres being on a quadric; that the five quadrics thus associated with the cyclide are confocal; and that any two of the five fixed spheres S cut one another orthogonally. When he announced his discovery of the orthogonal system of cyclides, Moutard remarked that for all the cyclides of the system these five spheres and five quadrics are the same. Making use of this result Darboux showed* that the equation of the system can be written in the form

$$\sum_{i=1}^{5} \frac{\left(\frac{S_i}{R_i}\right)^2}{\lambda - a_i} = 0, \tag{4}$$

where S_i is the power of the point with respect to the sphere S_i, R_i is its radius, a_i is a constant and λ the parameter of the system. This equation being necessarily of the third order the coefficient of λ^4 is identically zero. Hence we have the

* Sur une Classe remarquable, etc., p. 134. The fourth and fifth parts of this treatise are devoted to an exposition of the geometry of cyclides.

identity

$$\sum_{i=1}^{5}\left(\frac{S_i}{R_i}\right)^2 = 0 \tag{5}$$

for any point in space. Darboux saw in these quantities S_i/R_i, subject to the above condition, a new type of coordinate which could be used to good purpose in certain problems. Thus were discovered *pentaspherical coordinates*. An important theorem involving them is the following:

If five particular solutions $x_1, \cdots, x_5$ of an equation of the form

$$\frac{\partial^2\theta}{\partial u\partial v} = A\frac{\partial\theta}{\partial u} + B\frac{\partial\theta}{\partial v} + C\theta$$

satisfy the relation $\Sigma x_i^2 = 0$, the quantities x_i are the pentaspherical coordinates of a surface upon which the parametric curves are the lines of curvature. When, in particular, the above equation is reducible to the form (3), the surface is isothermic. Important consequences of this result have been obtained by Darboux and Guichard.

From the equations of Gauss it follows that the nonhomogeneous coordinates x, y, z of a surface, referred to a conjugate system of curves $u = \text{const.}$, $v = \text{const.}$, satisfy an equation of the Laplace form

$$\frac{\partial^2\theta}{\partial u\partial v} = a\frac{\partial\theta}{\partial u} + b\frac{\partial\theta}{\partial v}, \tag{6}$$

called the point equation of the system. However, Darboux called attention to the fact that each function $\varphi(x, y, z)$, in general, determines a conjugate system, found by the solution of an ordinary differential equation of the first order and second degree, in the sense that the corresponding equation (6) admits the solution φ as well as x, y, and z. In particular, the lines of curvature of any surface are characterized by the property that they form the conjugate system for which $\varphi \equiv x^2 + y^2 + z^2$. Guichard has extended this conception and considered the remarkable class of conjugate systems n, O, defined by the property that the point equation of such a system admits $n - 1$ solutions $t_1, \cdots, t_{n-1}$ and that $x^2 + y^2 + z^2 - t_1^2 - \cdots - t_{n-1}^2$ also is a solution.

Darboux has been such a conspicuous worker in differential geometry that one may make the mistake of thinking that his

geometrical work has been confined to this field. To offset this impression we recall that the greater part of his treatment of cyclics and cyclides is not differential. We mention also his studies of Poncelet polygons, and their generalization to polygons inscribed and circumscribed to ellipsoids, and some of his researches on the wave surface. It is an interesting fact that his first treatise, previously referred to, and his last, Principes de Géométrie analytique, just published,* deal almost entirely with finite geometry.

In 1868 Beltrami published† his striking interpretation of the geometry of Lobachevsky, Bolyai and Gauss in terms of geodesics on a pseudospherical surface. Three years later Klein‡ pointed out the significance for non-euclidean geometry of the concept of an absolute and of the associated metric which Cayley had developed in his Sixth Memoir on Quantics, twelve years before. These new ideas made a decided impression on Darboux. He made use of the Cayley measurement in the theory of cyclides as presented in his first treatise, and in a note appended to this volume he studied geodesics and lines of curvature in Cayleyan geometry. These ideas are developed more fully in the last chapter of the third volume of his Leçons, and are applied there to the determination of surfaces upon which there is a conjugate system of curves all of whose tangents are tangent to a quadric also. One of the five parts of the Principes de Géométrie analytique is devoted to a very clear exposition of Cayleyan geometry, displacements in this geometry and its trigonometry. But for one reason or another Darboux confined his researches to euclidean space, and he had no part in the development of the theory of non-euclidean differential geometry.

Darboux was a strong advocate of the use of imaginary elements in the study of geometry. He believed that their use was as necessary in geometry as in analysis. He had been impressed by the success with which they had been employed in the solution of the problem of minimal surfaces. From the very beginning he made use in his papers of the isotropic line, the null sphere (the isotropic cone) and the general isotropic developable. In his first memoir on orthogonal systems of

* This treatise is in fact an enlarged edition of the first. The writer will shortly publish a detailed review of it.

† *Giornale di Mathematiche*, vol. 6 (1868), pp. 284–312.

‡ "Ueber die sogennante Nicht-Euklidische Geometrie," *Math. Annalen*, vol. 4 (1871), pp. 573–625.

surfaces he showed that the envelope of the surfaces of such a system, when defined by a single equation, is an isotropic developable. We have given examples of the use of these elements in the theorem cited for rolling surfaces, and likewise in the solution of the problem of spherical representation. Another striking example is afforded by the following theorem, which gives a general method of obtaining surfaces with plane lines of curvature in one system: If a developable D rolls over a developable D_1 in such a way that all the points of a generator are in contact at any instant, an isotropic developable invariably fixed with respect to D will be cut by successive planes of contact along lines of curvature of the surface which is their locus.

In a discussion of so prolific a writer as Darboux one can merely mention the results of some of his investigations and cannot give an idea of the details. However, it would be a mistake in his case not to refer to the character of the methods used. Darboux gives to Combescure the credit of being the first to apply the considerations of kinematics to the study of the theory of surfaces with the consequent use of moving coordinate axes. But to Darboux we are indebted for a realization of the power of this method, and for its systematic development and exposition. This exposition is to be found in the first two volumes of his Leçons, with applications later to the discussion of particular types of surfaces and of orthogonal systems. The method has been used extensively by his students and followers, and it is sure to be used by any worker in the field of differential geometry who is familiar with its processes.

Darboux's ability was based on a rare combination of geometrical fancy and analytical power. He did not sympathize with those who use only geometrical reasoning in attacking geometrical problems, nor with those who feel that there is a certain virtue in adhering strictly to analytical processes. His geometrical proofs of the theorems dealing with rolling surfaces referred to above are as pure as they are simple and beautiful. No less brilliant are his reductions of various geometrical problems to a common analytical basis, and their solution and development from a common point of view. At times in the solution of a problem we find a combination of the two methods. This is seen in various parts of his treatises, and in particular in the memoir in which he solved the problem of finding the two-parameter families of spheres for

which the correspondence between the two sheets of the envelope is conformal.* By geometrical considerations it is shown that the lines of curvature on the two sheets are in correspondence, and then by analysis that they are isothermic surfaces. Here is established the transformations of isothermic surfaces, which Bianchi has called the transformations D_m, and the associated deformations of quadrics. However, we believe that Darboux's interest lay rather in analytical geometry. This was natural in one who possessed such remarkable powers as an analyst. We have not the time to discuss, nor indeed to enumerate, the contributions to analysis which Darboux made while in pursuit of the solution of geometrical problems. Surely we find realized in his works his belief that "the alliance between geometry and analysis is useful and productive; perhaps this alliance is a condition for the success of both."

In the foregoing sections we have tried to give an idea of the scope of Darboux's contribution to geometry. His writings possess not only content but singular finish and refinement of style. In the presentation of results the form of exposition was carefully studied. Darboux's varied powers combined with his personality in making him a great teacher, so that he always had about him a group of able students. In common with Monge he was not content with discoveries, but felt that it was equally important to make disciples. Like this distinguished predecessor he developed a large group of geometers, including Guichard, Koenigs, Cosserat, Demoulin, Tzitzeica, and Demartres. Their brilliant researches are the best tribute to his teaching. His spirit will continue to live in these men, and in the many who will turn to his works for direction and inspiration. Hence of Darboux also the prophecy may be true which Lagrange made of Monge, when he said "With his geometry this devil of a man will make himself immortal."

PRINCETON UNIVERSITY.

* "Sur les surfaces isothermiques," *Annales de l'Ecole Normale*, ser. 3, vol. 16 (1899), pp. 491–508.

CREMONA'S WORKS.

Opere matematiche di Luigi Cremona. Pubblicate sotto gli auspici della R. Accademia dei Lincei. Volumes 1 and 2. Milano, Hoepli, 1914 and 1915. Quarto, viii + 492, 451 pp.

CREMONA'S career as geometer and teacher covered very nearly the second half of the nineteenth century. Born at Pavia in 1830, he was only eighteen when his ardent patriotism drew him into the war of independence. Returning after a year and a half, at the close of the war, he studied at the University of Pavia under Bordoni and Casorati and in 1853 took the laureate in civil engineering and architecture. After seven years of teaching in lower schools, he was called in 1860 to the University of Bologna as first professor of projective geometry and mechanics. After six years of intense activity there, he returned to Milan as a colleague of Brioschi at the Polytechnic and Normal School, training teachers in graphical statics for the technical institutes of the new Italy. From 1873 until the end of his life, 1903, he lectured in the University of Rome, on geometry, graphical statics, and "higher mathematics," meanwhile giving time and care without stint to the school of engineering, of which he was the founder and director. Usually also he gave courses in the normal department, to which he attached no less importance than to the more purely theoretical studies.

It may be doubted whether any great teacher has been actuated primarily by considerations of economic utility. While Cremona was intensely patriotic, it is evident from his writings that it was the innate love of his chosen science that moved him to teaching and to the preparation of the books through which his name is most widely known. In this collection the editors have not included his Elements of Projective Geometry; but we have his Introduction to a geometrical Theory of Plane Curves (1862), which was later translated (1865) into German by Curtze, and attained wide circulation and use; also his Fundamentals of a geometric Theory of Surfaces. This latter, combined with the memoir on Cubic Surfaces, was also most extensively known in Curtze's German version (1869). Here too are included, of course, the two epoch-making essays on transformations of

plane curves (1863, 1865). It will be noticed that all these appeared during the Bologna period of six years. By way of comparison we note that Salmon's Higher Plane Curves first appeared in 1852, and that most of Steiner's work on algebraic curves was published between 1848 and 1854.

These two volumes* contain 78 titles, covering the author's publications down to 1868, from Pavia, Milan, and Bologna. Of these 78, 45 are dated from Bologna. Nine in the first, seven in the second volume, are answers to questions proposed in the *Nouvelles Annales*. Eight are book reviews, and neither dull nor devoid of significance; four are historical articles or addresses intended to stimulate geometrical study and research. One should read first of all the short peroration (volume 1, pages 252–3) of the inaugural address to his course on higher geometry at Bologna. Here occurs the often quoted climactic injunction: "Credite all' avvenire! questa è la religione del nostro secolo." But of more value to the young mathematician is the admonition: "Senza un' incrollabile costanza nella fatica non si giunge a possedere una scienza. Se questo nobile proposito è in voi, io vi dico che la scienza vi apparirà bella e ammiranda, e voi l'amerete così fortemente che allora in poi gli studi intensi vi riusciranno una dolce necessità della vita." There are not many such valuable sources of information and inspiration accessible to ambitious students.

Chasles was the type on whom at first Cremona modeled his own work. The geometrical purism of von Staudt was a later influence. Hence it is natural to find metric foundations for geometric definitions instead of, or interchangeably with, projective. The apparatus of algebraic geometry is built upon polars, and these upon distances. The geometric method is principally a use of terms or descriptive relations instead of equations. And the beginnings of enumerative geometry are here: we find questions of intersections or determining conditions for curves or surfaces treated by what has since become the principle of the Erhaltung der Anzahl, the axiom that the number persists even when the loci involved become specialized or degenerate. What Cremona is trying to attain and to diffuse is, not a critical knowledge of foundations, but an extensive knowledge of objects and theorems in projective geometry. This will doubtless be the

* The third and concluding volume has been published more recently (1917).

path followed by students for a long time to come; for it is easier to analyze and to discriminate than to unite discrete theories.

The Introduction contains an account quite complete, at its date, of plane cubic curves. It was to be desired, as Curtze said, that the theory of surfaces should cover those of the third order, and for this there was ample material in Cremona's Steiner prize essay. The depiction of a general cubic surface upon a plane, its rationality, was discovered independently by Cremona and Clebsch, and enabled the former to work out fully the geometry of algebraic curves upon the surface. This theme is more elaborately studied by R. Sturm in his well-known paper in the *Mathematische Annalen*, volume 21 (1883). A footnote there might suggest the notion that Cremona had committed a serious error in mentioning only one out of nine possible cases where the intersection of a cubic with a sextic surface breaks up into two twisted nonic curves. A reference to Cremona's own statement shows, however, that he was not attempting an exhaustive list of cases. This particular statement, like many in the same chapter, concludes with "etc., etc." That derogatory criticism was not in Sturm's intention is apparent also from the opening section of his "Nachruf" for his friend (*Archiv für Mathematik und Physik*, series 3, volume 8, 1904, pages 11–29).

In this Nachruf and in two others, Noether's in *Mathematische Annalen*, volume 59 (1904), pages 1–19, and Bertini's in *Proceedings of the London Mathematical Society*, series 2, volume 1 (1903–4), pages V–XVIII, is found a clear and careful analysis of most of Cremona's important scientific papers. In particular, Sturm very neatly and fully analyzes the series of papers on twisted cubic curves, ten in number, which appeared from 1858 to 1864. The first two were devoted mainly to proving theorems, some twenty-five in number, which Chasles had published without proof in a note to his Aperçu historique, but both these and the later ones contain much that was original. The three that Chasles gives upon loci arising in the infinitesimal motion of a rigid body Cremona seems not to consider important enough to reproduce, as indeed they are but restatements of abstract theorems. The erroneous theorem of Chasles he does not criticize nor correct, namely that the locus of vertices of quadric cones passing through six arbitrarily fixed points in space is a twisted curve

of the third order, whereas it is obviously two-dimensional and actually a Weddle's quartic surface.

One of these papers on twisted cubics had the experience, not unusual before the advent of the *Jahrbuch über die Fortschritte der Mathematik*, of being forgotten, and later duplicated. Böklen and F. Meyer in 1884 studied the question as new: When has a cubical parabola a directrix? If every osculating plane is perpendicular to two others that meet at a right angle, the locus of their common point is termed a directrix. When there is a directrix, what is its nature? The cubical parabola is a twisted cubic which osculates the plane at infinity, so that its other osculating planes trace out at infinity a conic. Orthogonality is conjugateness with respect to the absolute, an imaginary circle at infinity; so that the question concerns polar triangles of one conic circumscribed to another. These matters, and others related to them, had been thoroughly discussed in Cremona's final paper on twisted cubics in 1864, No. 50 of the present collection, particularly in § V. This fact probably escaped the notice even of Schroeter, whose writings were contemporaneous; for he does not mention it in his highly interesting paper: "Metrische Eigenschaften der cubischen Parabel," in *Mathematische Annalen*, volume 25 (1885). Such instances as this, of truths discovered yet insecurely fixed in the body of science, help one to realize the value of the Encyklopädie der mathematischen Wissenschaften.

Cremona himself was conscientious and indefatigable in searching out the work of his predecessors upon matters that he himself was investigating. Witness for example the voluminous list cited in his review of the translation of Amiot's geometry. This virtue is not too common, and we may be permitted to quote from Poncelet, who published with pardonable pride in volume 1 of his Traité (page 420) a letter from Dupin containing this tribute to that kind of honest dealing. "Je vous loue beaucoup d'avoir aussi rappelé honorablement les travaux de tous vos prédécesseurs; croyez-moi, cela n'ôte rien à votre mérite, et donne une haute idée de votre caractère. Vous prouvez par là que vous n'avez rien de commun avec cette école égoïste qui voudrait faire un monopole de la célébrité mathématique. Qu' ils maigrissent à leur gré de l'embonpoint d'autrui, et faites-les maigrir." Apropos, one reads with admiration the footnote

(volume 2, page 18) wherein Cremona, after two pages of historical summary, excuses his previous ignorance of, and failure to cite, the works of Möbius and Seydewitz to which an essay of Schroeter had now directed his attention. He concludes with the phrase: "A présent je restitue unicuique suum." To be just is not easy, but he believed it to be a duty.

Of all these valuable works which constitute the visible monument of a valiant geometrician, those which will longest secure his fame are those numbered 40 and 62 in volume 2, upon geometric transformations of plane figures. Beyond the linear or projective transformations of the plane there were known the quadric inversions of Magnus, changing lines into conics through three fundamental points and those exceptional points into singular lines, to be discarded. Cremona described at once the highest generalization of these transformations, one-to-one for all points of the plane except a finite set of fundamental points. He found that it must be mediated by a net of rational curves; any two intersecting in one variable point, and in fixed points, ordinary or multiple, which are the fundamental points and which are themselves transformed into singular rational curves of the same orders as the indices of multiplicity of the points. When the fundamental points are enumerated by classes according to their several indices, the set of class numbers for the inverse transformation is found to be the same as for the direct, but usually related to different indices. Tables of such rational nets of low orders were made out by Cremona and Cayley, and a wide new vista seemed opening (such indeed it was and is) when simultaneously three investigators announced that the most general Cremona transformation is equivalent to a succession of quadric transformations of Magnus's type. This seemed a climax, and a set-back to certain expectations. But the fact remained, that elements do not constitute a theory, and that the generators of a group are no more important than its invariants. These invariants, for the most part, are yet to be determined, both synthetically and algebraically.

The expectation of geometrical inventors was turned next upon space of three dimensions; and there Cremona showed how a great variety of particular transformations can be constructed, but anything like a general theory is still in the future. When found, if within this century, such one-to-one transformations of three-dimensional space are certain to be hailed as Cremona transformations.

It is honorable alike to the Royal Academy of the Lincei and to the colleagues of Cremona and younger mathematicians, that they unite to preserve in worthy form the works of a justly celebrated scientist and leader. Eighteen already have shared the not inconsiderable labor of thorough editing, and their corrections and explanatory notes, appended to each volume, form a valuable aid to the reader. The highest tribute that can be paid to the memory of a scientist is the labor that makes his work more useful to the next generation.

HENRY S. WHITE.

BLICHFELDT'S COLLINEATION GROUPS.

Finite Collineation Groups. By H. F. BLICHFELDT. Chicago, University of Chicago Press, 1917. 12mo. 12 + 194 pp.

THIS little volume forms a notable contribution to the series of mathematical texts by American authors that have appeared in recent years. Coming from the pen of an author who has an unusual mastery of his subject, it is moreover almost unique in its field, the promised text by Wiman for the Teubner series (as far as the reviewer is aware) not having appeared. Certain parts of the subject, particularly the theorems depending on the invariance of a Hermitian form and the theory of group characteristics, may be found in the second edition of Burnside's Theory of Groups, which appeared in 1911. A considerable part also of the material in the present treatise may be found in Part II of Finite Groups, by Miller, Blichfeldt, and Dickson, which was written by the same author.

On the other hand there is much in the present volume that cannot be found elsewhere except in scattered journal articles, and some of the results at the close of Chapter IV seem to be entirely new. The author's own share in the development of the subject is a very notable one, the theorems in Chapter IV concerning the linear groups in n variables being almost entirely his own. In addition the complete determination of the groups in three and four variables was first made by him, the earlier work along this line being reproduced in Chapters V and VII in a somewhat revised form. There are

in addition chapters on the "Elementary Properties of Linear Groups," "Groups of Operators and Substitution Groups," "The Linear Groups in Two Variables," "The Theory of Group Characteristics," and "The History and Applications of Linear Groups."

As the author remarks in the closing chapter, the theory of linear groups may be said to have originated with Klein, who was mainly interested in their application to the solution of algebraic equations. At about the same time, however, they were employed by several writers, notably Schwarz, Fuchs, and Jordan, in the study of linear differential equations having algebraic integrals. The theory received in this way such an impetus that it has now grown to considerable proportions.

The problem of the determination of the finite collineation groups has been attacked in two ways. One may ask either "What groups may be represented in a given number of variables?" or "In how many ways may a given abstract group be represented as a linear group?" Concerning the first of these questions there is the theorem of Jordan that the order of a finite linear group in n variables is of the form λf, where f is the order of an abelian self-conjugate subgroup, and where λ is inferior to a fixed number that depends only upon n. The theorems of Chapter IV enable the author to obtain a definite limit for λ; in fact, he was the first to obtain such a limit, although other (much higher) limits have been obtained by Bieberbach and Frobenius. The exact values that λ may have are known only for $n = 2, 3, 4$.

The theory of characteristics, which is due mainly to Frobenius, throws considerable light on the second question. One of the theorems obtained by this means is to the effect that if a given abstract group of order g be represented as a "regular" group of permutations on g variables, and then by means of a change of variables the group be represented by means of a series of "transitive" component groups, every possible representation of the given abstract group as a transitive linear group will occur in this series and the number of times it occurs is equal to the number of variables in which it is represented. Moreover, the number of such representations that are "non-equivalent" is equal to the total number of sets of conjugate operators of the given abstract group.

The author includes these theorems in his chapter on the

theory of group characteristics. It seems unfortunate, however, that he should have used the terms "equivalent or non-equivalent groups" instead of "equivalent or non-equivalent representations of a group." This point will be discussed more in detail later.

There seems no doubt that the author was desirous of developing the subject of linear groups proper as far as space would permit, and for this reason has omitted any extended discussion of the geometrical properties of the individual groups, such as their invariants. The methods used in the determination of the groups in three and four variables are as a rule analytical, although geometrical arguments have been employed in a few places.

In spite of the fact that the treatment was evidently not intended to be exhaustive, the reviewer was somewhat disappointed not to find anything concerning the groups that are known to exist in more than four variables. Perhaps the most interesting groups of this sort are the three systems in p^m, $(p^m - 1)/2$, and $(p^m + 1)/2$ variables that are isomorphic with the "abelian linear group" on $2m$ indices, the coefficients of which are residues, modulo p. For $m = 1, 2$ these groups have been considered by Klein and several others in their relation to elliptic and hyperelliptic functions. The ternary groups of order 216, 168, 60 and the quaternary groups of order 11520, 25920, 168 each belong to one of these systems. There are in addition groups in six, seven, and eight variables, reference to which might have been included among the historical notes in the closing chapter.

In taking up the consideration of the various chapters somewhat more in detail, special mention should be made of the fundamental theorem, due to several authors, that every finite linear group in n variables has an invariant definite Hermitian form. This theorem, which is included among those given in the first chapter, is used to establish the result, due originally to Maschke and later extended by Loewy, that every "reducible" group is "intransitive." Thus, for example, every collineation group in three-dimensional space that has an invariant plane must also leave fixed a point not in that plane. This result is used in turn to show that every linear group in which the transformations are commutative may be written in canonical form. It might perhaps have been well to direct the reader's attention to the fact that this is not necessarily the case for collineations.

The second chapter contains an introduction to the theories of abstract groups and permutation groups, introduced mainly for the benefit of readers who may have had no previous acquaintance with the subject. Under the former heading we find for example Sylow's theorem and some discussion of abelian groups and groups having for order a power of a prime. With regard to permutation groups there is included something concerning the representation of a given abstract group as a "regular" group and some theorems on the alternating and symmetric groups.

For the determination of the linear groups in two variables the author has chosen on account of its historical and geometrical interest the process employed by Klein, whereby the possible groups are shown to be isomorphic with the groups of rotations of the regular solids. An outline is also given of Jordan's process for finding these groups by means of a diophantine equation. Another somewhat similar equation which would answer the same purpose may be obtained from Theorem 4 of the chapter on group characteristics. It follows from this that the order g of any transitive collineation group in two variables must satisfy the equation

$$g = 4 + g\sum_i f_i(n_i - 2)/n_i,$$

where the summation is to be taken over the orders $n_1, n_2, \cdots$ of the cyclic subgroups in the different conjugate sets, and each $f_i = 1, 2$.

The fourth chapter is the one that is likely to attract the widest attention. One of the most interesting theorems that it contains is to the effect that the order of a primitive group in n variables cannot be divisible by a prime greater than $(n-1)(2n+1)$, a result first published by the author in the *Transactions* in 1903. Whether for large values of n there exist groups whose orders are divisible by primes nearly as large as this seems rather doubtful; in fact, as far as the reviewer is aware, no primitive group is known to exist whose order is divisible by a prime greater than $2n+1$. It is a real achievement, however, to have been able to obtain any limit at all.

In the analysis employed in the proof a certain equation involving only roots of unity is shown to exist connecting the characteristics of certain transformations. By use of Kron-

ecker's theorem concerning the irreducibility of the general cyclotomic equation the author ingeniously transforms this equation into a congruence, from which it follows that the product of any two transformations of order p, a prime exceeding the limit given above, must be either of order p or identity. For special values of n he is able to reduce this limit. For $n = 3, 4$ it is found that no primes exceeding 7, 13 respectively can divide the order of a primitive group.

The author's statement of Kronecker's theorem (§133), or rather the consequence that he deduces from it, is not quite accurate. For example, the rational equation of lowest degree that is satisfied by θ, a primitive 15th root of unity, is

$$\theta^8 - \theta^7 + \theta^5 - \theta^4 + \theta^3 - \theta + 1 = 0,$$

but the seven roots of unity appearing in this sum cannot be separated into sets of the sort that he describes. This does not form a serious defect in his argument, however, since it does follow from Kronecker's theorem that every sum of roots of unity that is equal to zero may be thrown into the form (4), page 85, by the addition and subtraction of roots that cancel each other by pairs. The second paragraph under 7°, §133, is subject to a similar modification.

There are also theorems that limit the conditions under which a primitive simple group can contain transformations having for order a power of a prime and commutative transformations of different orders. In Theorem 7, page 93, condition (B) is unnecessary, since there do not exist more than m different roots of unity whose mth powers are all equal to the same root.

In the latter part of the chapter an interesting type of analysis first employed by Bieberbach and Frobenius is used by the author to show that no primitive group can contain a transformation whose multipliers when represented graphically on the unit circle all lie on an arc that does not extend more than 60° on either side of some one of them. This theorem is found useful in the chapters on the groups in three and four variables. It is also used in the section that immediately follows to establish an upper limit for the order of any abelian subgroup that can be contained in a primitive group in n variables.

Finally by use of the various theorems of the chapter a limit is obtained for the order of a primitive group in n variables,

a result that has been referred to above. This limit is lower (as far as the reviewer is aware) than any that have been published previously. The two theorems moreover that were referred to in the preceding paragraph seem also to be new, although the first of these bears a close relation to one obtained by Frobenius.

In the succeeding chapter these results are applied to the determination of the primitive groups in three variables. The first accurate solution of this problem was obtained by the author of this text, the previous attempts by Jordan and Valentiner being only partially successful. These groups are found to be the "Hessian" group of order 216, so called by Jordan on account of its relation to the inflexional points of a pencil of cubic curves, two of its subgroups of orders 72 and 36, and three simple groups of order 60, 168, 360. The last two were first found in the order named by Klein and Valentiner.

Although a number of expositions of the theory of group characteristics have been given, notably by Frobenius, Burnside, Schur, and Dickson, the author does not follow closely any of them. In making essential use of the invariance of a Hermitian form his treatment probably resembles Burnside's more than any of the others.

Some of the theorems here give additional information concerning the possible primitive groups in n variables. By means of one of them it follows that if any group contains transformations of orders p and q, two distinct primes each greater than $n + 1$, it must also contain transformations of order pq. As far as the reviewer is aware, no primitive group of this sort is known to exist. Another useful relation is that for a transitive group the sum of the products of the characteristics and their conjugate imaginaries (for each of the transformations) is equal to the order of the group. An application of this theorem to the determination of the binary groups has been mentioned above.

Some of the theorems concerning the representation of a given abstract group as a linear group have already been referred to. To illustrate the sense in which the author uses the terms "equivalent or non-equivalent groups," we consider the linear group of order 27 generated by $x' = y$, $y' = z$, $z' = x$, and $x' = x$, $y' = \omega y$, $z' = \omega^2 z$, where $\omega^2 + \omega + 1 = 0$. We may put the transformations of this group into (1, 1)

correspondence with those obtained by replacing ω by ω^2, which latter form therefore a group simply isomorphic with the first. According to the author's terminology (Theorem 5, page 128) these two groups are "non-equivalent," doubtless for the reason that it is impossible by a change of variables to make the *corresponding* transformations simultaneously identical. On the other hand they contain exactly the same transformations in the form in which they are written, though arranged in a different order.

This seems to the reviewer an unusual use of these terms. As the author apparently does not mean the definition on page 64 to be taken in this sense, it seems not unlikely that the reader may place a wrong interpretation on some of the theorems in this chapter. That this is the meaning that must be attached to the word "equivalent" may be seen from a careful examination of the proof of Theorem 5.

An interesting application of the theory of characteristics is made at the close of the chapter to prove the well known theorem that every group of order $p^a q^b$ is composite.

In Chapter VII the author applies the theorems of Chapters IV and VI to the determination of the primitive groups in four variables. The exposition of the subject in this treatise has been somewhat modified from the original one, which appeared in the *Mathematische Annalen* in 1905, and which was an achievement of no mean order. If any primitive group is not itself simple (as a collineation group), then somewhere in its chain of factor groups there will occur either a primitive simple group or else a group that is not primitive, thus affording a basis for a classification.

It follows at once from geometrical considerations and by use of the knowledge concerning the groups in two and three variables that no primitive group in four variables can contain transformations with two pairs of equal multipliers of higher order than 5 or transformations with three equal multipliers (homologies) of higher order than 3. In a separate discussion it is shown that transformations of the former type having the order 5 may also be ruled out. The simple group of order 25920 is found to be the only primitive group that contains homologies of order 3.

The discussion at the bottom of page 145 may not be entirely clear to a reader. The author remarks that the two binary groups of order $60 \cdot 2$ must be "equivalent," since the two

generating transformations of order 5 have the same multipliers in the two cases, the term being used doubtless in the same sense as in the preceding chapter. It does not seem evident without further consideration that the quaternary group might not be (2, 1) isomorphic with either of the two binary groups, in which case Theorem 5 of Chapter VI would not give the desired result. This is apparently the theorem that the author is using here, although it is not explicitly referred to.

The proof of Theorem 3 would have been much simplified if after establishing the existence of a subgroup of order 648 the author had used the fact that any two homologies of order 3 that are not commutative must generate a group of order 24 in which all four homologies are conjugate and which contains an invariant reflection. Thus a homology must be conjugate with any that are not commutative with it and hence in a primitive group with all others. Also any homology not in the subgroup of order 648 must be commutative with one of the nine reflections that this subgroup contains, from which it follows that there cannot be more than 40 homologies altogether. By a proper choice of coordinates one of the nine reflections commutative with "D" may be taken as "F."

In view of Maschke's theorem (page 23) and the fact that there is no larger collineation group in three variables containing a G_{216}, the argument on page 151 beginning, "We now write down, etc.," appears superfluous.

Having eliminated from consideration certain types of transformations the author is able to show that the possible Sylow subgroups that may occur in a primitive simple group are of a comparatively limited number of types. A number of the theorems of Chapters IV and VI are found useful. One device that is employed consists of the introduction of line coordinates and the application of some of these theorems to the resulting group in six variables. In the reviewer's opinion it would have been desirable to have given some explanation of the method of excluding transformations of order 9. The final conclusion is that the only primitive simple groups that exist are of order 60 (two types), 168, 360, 2520, 25920.

The only groups in this list that can be contained self-conjugately in larger groups are those that are isomorphic with the alternating groups on five or six letters. Each of

these is contained by a group of twice its order isomorphic with the corresponding symmetric group. These in turn are not contained self-conjugately by any larger groups.

The primitive groups containing invariant intransitive subgroups are all found to leave invariant a quadric surface. Some of these can be enlarged by the addition of transformations that interchange the two sets of rulings. Finally it is shown that the only primitive groups that contain imprimitive invariant subgroups are the one of order 11520 and certain of its subgroups.

The final chapter on the history and applications of linear groups has been referred to above. The author gives an interesting sketch of the methods used in their application to the solution of algebraic equations and describes also their relation to linear differential equations having algebraic integrals.

The book appears in a neat, attractive form and the proof-reading has been carefully done, only a few errata being found. Those that were noticed, together with some minor comments, are as follows:

P. 19 (next to the last line). Replace $\bar{x}_{n-1}$ by $\bar{x}_n$.
P. 20 (8th line). Replace one y_{n-1} by $\bar{y}_{n-1}$.
P. 20 (19th line). Replace f'_θ by f.
P. 65 (next to last line). The value of Y should be $Y = x_1\bar{x}_2 + x_2\bar{x}_1$.
P. 72. The numbers given here are the numbers of *rotations* rather than the numbers of the *axes*. There are only 10 different axes of period 3 and 6 of period 5.
P. 79 (7th line). Replace Y_2'' by Y_1''.
P. 90 (3d line from bottom). Replace $p_n \equiv kp$ by $p^n \geqq kp$.
P. 100. It is perhaps worth while to remark that the sets of intransitivity "of highest index" referred to at the middle of the page may not coincide with the "ultimate" sets of intransitivity referred to in the preceding discussion.
P. 110. The author apparently intended that T' should be of period 4. For this we may take $\alpha = 1$, $\beta\gamma = -1$.
P. 112 (6th line). Replace $2^3 \cdot 3^2 \cdot \phi$ by $2^3 \cdot 3^2$.
P. 113 (last line). A factor 1/7 has been omitted from the first expression for h.
P. 126 (middle). The expression in the second parenthesis should be $Y_2 + a_2 Y_3$.
P. 154. The orders of Q for the groups (g), (j), (k) should be respectively 10ϕ, 75ϕ, 150ϕ.

In view of the unusual conditions prevailing at the present time the book will undoubtedly not attract the attention it deserves. It ought to serve however to stimulate interest in the subject at least in this country. There are few, if any, theories that are possessed of greater elegance or that offer a more direct challenge to the mathematical investigator.

As a text for class-room use it will be found very suitable except perhaps for those whose interests center mainly in the geometrical aspects of the subject. Some of the analysis in Chapter IV may possibly be found rather difficult by immature students, but by suitable omissions no trouble would be experienced. Not only the author but the publishers as well are to be congratulated on their part in the production of the book.

HOWARD H. MITCHELL.

SHORTER NOTICES.

Elliptic Integrals. By HARRIS HANCOCK. New York, Wiley, 1917. 104 pp.

THIS volume is number 18 of the well-known series of Mathematical Monographs edited by Mansfield Merriman and Robert S. Woodward. It was prepared in response to a request from the editors for a work of about one hundred octavo pages on elliptic integrals which should "relate almost entirely to the three well-known elliptic integrals, with tables and examples showing practical applications." The monograph is confined to the Legendre-Jacobi theory and the discussion is limited almost entirely to the elliptic integrals of the first and second kinds.

After a short introduction (pages 5–8), mostly historical, there follows in Chapter I (pages 9–23) an elementary discussion of the three kinds of elliptic integrals and the Legendrian transformations. The Jacobi elliptic functions are treated in Chapter II (pages 24–40). Chapter III (pages 41–64) is devoted to elliptic integrals of the first kind and Chapter IV (pages 65–87) to numerical computation of the elliptic integrals of the first and second kinds and to Landen's transformations. Several miscellaneous examples and problems are given in Chapter V (pages 88–91). In the sixth and last chapter (pages 92–101) are three five-place tables as follows: Table I, the complete integrals of the first and second kinds, page 93; Table II, elliptic integrals of the first kind, pages 94–97; Table III, elliptic integrals of the second kind, pages 99–101.

A few misprints should be mentioned: On page 6, line 4, "*Application*" is printed for "*Applications*"; on page 12, line 6 below, "within" is printed for "in"; on page 24, second exposed line, supply lower limit of integration 0, and in the third line below this replace "written" by "wrote"; at the middle of page 26, supply lower limit of integration 0; on page 30, line 4, replace "real" by "pure imaginary"; on page 32, line 3 below the figure, "rectangles" is preferable to "parallelograms."

There is lack of consistency in the use of a_0 on page 10. It gives a jolt to the reviewer to read (page 20, lines 6 and 5 below) "F_1 increases from $\frac{1}{2}\pi$ to logarithmic infinity." There is no difference between the infinities approached by $\log x$ and x itself; the difference is in the way in which this infinity is approached. It is not clear how one may "observe" the transcendental nature of K and K' by considering the series exhibited on page 26. The grammatical connections in the author's sentences are not always felicitous, as one may verify (for examples) by noticing the sentence beginning near the foot of page 59 and that beginning near the foot of page 76.

We have now done our worst in the criticism of this monograph. It has many merits to commend itself to our interest. It affords in small compass an introduction to certain portions of the theory of elliptic integrals and functions, portions which in this concise exposition will be useful to a considerable number of individuals. It certainly covers very well just the ground which the editors asked the author to cover.

Perhaps it is never proper for a reviewer to quarrel with the author for not having written a different book from the one which he did write, especially when he has carried out a request from the editors of a series. But, in the present instance, one can hardly avoid raising the question as to which is likely to be more useful now, a hundred-page monograph on elliptic integrals such as the one under consideration or a like monograph on elliptic functions approaching them from the fascinating function-theoretic points of view. There seems little room to doubt that the latter would prove of more service and have a wider distribution. It is to be hoped, therefore, that the appearance of the monograph now before us will not prevent the preparation and publication of a similar one on elliptic functions.

R. D. CARMICHAEL.

The Continuum and Other Types of Serial Order. By EDWARD V. HUNTINGTON. Second edition. Harvard University Press, 1917. vii + 82 pp.

FROM some cause the writer of this notice had never read the first edition of this book. It was therefore with a fresh, as well as a peculiar, pleasure that he followed the author in his remarkably beautiful and satisfying exposition of one of the highly fascinating subjects of modern mathematics. Any one else not at first hand acquainted with this work, whether or not he has been interested in the theory of the continuum and related matters, will find awaiting him here a purely intellectual delight of unusual order. It would be hard to seek out anywhere a more satisfying account of a topic in mathematics.

It should be emphasized that this exposition is useful to others as well as to mathematicians. Such in fact is one of its leading values. It deals with a subject which requires no technical knowledge of mathematics and which therefore is "peculiarly accessible to the increasing number of non-mathematical students of scientific method who wish to keep in touch with recent developments in the logic of mathematics." In the treatment before us full use has been made of this characteristic of the subject. The mathematical prerequisites for understanding it have been reduced (except in one or two illustrative examples) to a knowledge of the natural numbers 1, 2, 3, $\cdots$, and the simplest facts of elementary geometry. Numerous well-chosen examples are given to illustrate in a concrete way the abstract notions which are treated. The work is therefore accessible to all mature persons who are interested in purely intellectual matters and take pleasure in the more ideal æsthetic elements of thought.

The mathematical reader does not lose by this non-technical exposition; on the other hand, such a treatment has for him a peculiar value of its own. One has only to run over a general list of mathematical topics for a given definite period, as, for instance, the six months covered by a single recent issue of the *Revue Semestrielle*, to get a striking realization of the diversity of elements in modern mathematics and of the fact that mathematicians have not only largely grown apart from other scientific workers but that they have become, in small groups, isolated from one another and often are hardly able to speak across the chasm which separates one group from even its

nearest neighbor. A great disintegrating tendency arises from this fact; and there seems to be a growing feeling that it is necessary to take steps to combat it. Probably this can be done in no way better than by having non-technical developments of connected bodies of doctrine in many modern subjects. For such a treatment the exposition in Huntington's Continuum may possibly serve as an ideal toward which to labor, but nevertheless an ideal which probably can be attained only in relatively rare cases owing to the nature of the topics themselves.

R. D. CARMICHAEL.

A Course in Mathematical Analysis: Differential Equations, being Part II of Volume II. By EDOUARD GOURSAT. Translated by EARLE RAYMOND HEDRICK and OTTO DUNKEL. Boston, Ginn and Company, 1917. viii + 300 pp.

To the translators it seemed best, for the purposes of American schools, to issue separately the two parts of the second volume of Goursat's Cours d'Analyse Mathématique; and this has been done with the approval of Professor Goursat. The treatise before us consists of the second half of the second volume.

This work is too widely known for us to give here a statement as to its contents; it is too favorably known for us to attempt an analysis of its main characteristics. The excellent translation is worthy of the original. The printing and in fact the whole mechanical make-up are appropriate to the contents.

On page 109 insert "the" between the fourth and fifth words of line four. On page 115, line 4, "Appel" is printed for "Appell." "Mécanique" is spelled incorrectly in the last line of page 151.

The sentence in lines 3–5 of page 102 the reviewer would be pleased to see put in stronger terms. Concerning the differential equation

$$y^{(n)} + a_1 y^{(n-1)} + \cdots + a_n y + a_{n+1} = 0$$

it is said: "It may, however, happen that a point α is a singular point for some of the coefficients a_i without being a singular point for all the integrals." Here "all the integrals" might be replaced by "any of the integrals," as one sees from

the equation

$$y'' - \frac{2\sin x}{\sin x - \cos x}y' + \frac{\sin x + \cos x}{\sin x - \cos x}y = 0,$$

of which the general solution is

$$y = c_1 e^x + c_2 \sin x.$$

Here the coefficients have poles but the general solution is an entire function. Such equations of the second order are readily formed in unlimited number by determining each one so as to have two entire functions as particular integrals, these entire functions being chosen so that their real zeros do not separate each other. This fact is an immediate corollary of Sturm's zero-separation theorem.

R. D. CARMICHAEL.

Elementi di Aritmetica, con note storiche e numerose questioni varie per le scuole medie superiori, Parte prima: *Numeri interi—Operazioni, divisibilità, numeri primi.* (Third edition, Trimarchi, Palermo, 1916. vi + 134 pp. Fourth edition, 1918. 132 pp.) By Professor GAETANO FAZZARI, of Palermo. Price, L 1.60.

THIS arithmetic includes, as is common in European texts, much algebraic material. Thus discussion of such topics as the laws of commutation and association, and the euclidean process of finding G.C.D. appear. The fundamental operations of arithmetic are discussed both from the elementary point of view and from that of the higher mathematics.

The Hindu method of "multiplication in one line," by obtaining successively those products which contain units, tens, hundreds, $\cdots$, is explained. Division by use of the complement, frequently a convenient method, is given, illustrated by the division of 47830219 by 68947. The process is as follows, and may be regarded as division by 100000 − 31053.

$$\begin{array}{rl} & 31053 \\ 47830219 : & 68947 \\ 186318 & \overline{} \\ \hline & 693 \\ 6646201 & \\ 279477 & \\ \hline 9256789 & \\ 93159 & \\ \hline 349948 & \end{array}$$

Similar devices, more or less illuminating to the young student, are shown in connection with the discussion of all the fundamental operations. A prominent place is given to interesting material in elementary number theory.

For half a century Professor Fazzari has been a distinguished writer on the history of science. This interest is reflected in the well-chosen historical and philological notes which appear at the end of each chapter. The exercises in 31 pages at the end of the work are selected from ancient and modern sources to illustrate curious and interesting properties of numbers.

The fourth edition is, as the preface states, practically unchanged. On its face, this edition is extremely modern, as it is dated "Palermo, 1918" and the author's preface is dated "Palermo, September, 1918.

LOUIS C. KARPINSKI.

Lehrbuch der darstellenden Geometrie für technische Hochschulen. By EMIL MÜLLER, professor of mathematics at the technical school of Vienna. Second volume, final instalment. Leipzig, Teubner, 1916. 130–360 pp.; 141–328 figures.

THE first volume of Professor Müller's book was reviewed in the BULLETIN, volume 16, page 136, and the first instalment of the second volume in volume 20, page 258. The present part is concerned with oblique axonometry and central perspective. The former is founded on Polke's theorem, a number of alternate proofs of which are given. As in the preceding parts, the text is well supplied with historical and bibliographical foot-notes, emphasizing that the science was of slow, gradual, and international growth.

The treatment of general axonometric representation is so exhaustive as to make the book hardly suitable as a text, but it is all the more serviceable as a handbook. A full discussion of æsthetic advantages and disadvantages is included, and the method is critically compared with those met with in the earlier parts of the book. Between this subject and central perspective a chapter on oblique projection is inserted, with applications to circles, spheres, and surfaces of revolution. A goodly list of exercises follows each chapter.

The development of the principles of perspective is particularly clear and readable. It is first presented independently, then shown to be approachable also from the h, v drawings, or from axonometric ones. The mathematical standpoint

is never lost sight of, although no pains are spared to bring out desired artistic effects. The purely theoretical development is followed by a description of various devices to aid in drawing, including parallel rulers, Nicholson's centrolinead, Schilling's three-bar ruler, and a number of linkages. The final chapter contains an elaborate application of the method to the representation of curves and surfaces, including arches, trusses, etc.; a few pages are given to explaining the meaning and uses of photogrammetry.

The book is provided with a full index, and a list of all the authors cited. VIRGIL SNYDER.

The Method of Least Squares with Applications (third edition). By DANA P. BARTLETT. A. D. Maclachlan, 502 Boylston St., Boston, 1915. 143 + xi pp. Price $2.25.

FOR clear, concise statement in readable form and for systematic treatment of the whole range of ordinary applications of the method, we have not found the equal of this text by Professor Bartlett, of the Massachusetts Institute of Technology. These qualities have led the reviewer to adopt it for use covering this phase of the course with his students in "the mathematics of statistics."

The author has avoided making provision for special lines of work in which the method is applied, and so the instructor is free in the development when such special applications are needed. A well chosen example with complete solution follows every important development of the theory, but the 155 well graded examples are arranged like the punctuation marks in a certain school-boy's composition, which had none in the body of it but an ample supply of all sorts that might be needed at the close with the instruction "Put them in where you may wish."

The appendix gives the elements of the theory of probability, a bibliography, and three tables useful in applying the method. Throughout the text, references are made to more extended treatments in standard texts. CHARLES C. GROVE.

Text-Book of Mechanics. Vol. VI. *Thermodynamics.* By LOUIS A. MARTIN, JR. New York, Wiley, 1916.

THIS book, which is the sixth volume of a series of texts on mechanics, is a remarkably compact and comprehensive

theoretical treatment of the mechanics of gases and vapors.

The first part deals with the laws of ideal gases, including a very thorough discussion of the fundamental equation $pv = RT$ and its graphical representation on the pv plane, as well as the representation of polytropic curves on log-log paper.

The first of the applications which follow is a brief chapter on compressors. The various cycles of hot air and internal combustion engines are then analyzed, although in view of the increasing importance of the latter a great deal more space might well have been devoted to a discussion of the Otto and Diesel cycles, especially in relation to the conditions under which their efficiencies approach a maximum.

This first part then concludes with a chapter on the two fundamental laws of thermodynamics, namely the law of the conservation of energy and of the flow of heat, including a discussion of Carnot's principle and Kelvin's absolute scale of temperature, and lastly a chapter on entropy and temperature-entropy diagrams.

The second part on vapors follows practically the same order of treatment as that on gases. The general properties of vapors are first discussed, followed by the properties peculiar to dry, wet, and superheated vapors. This is followed in order by chapters on the entropy of vapors, changes of state of vapors, and vapor cycles.

The third part is essentially a supplement on the flow of fluids, including the effect of throttling, the use of Venturi meters, etc., while the last chapter takes up very briefly the fundamental differential equations of thermodynamics.

An important feature of the book is a collection of 382 numerical problems.

The book is well printed and illustrated, and carefully written, and altogether is an admirable elementary presentation of the theoretical side of the subject. This subject is of such importance, especially at the present time, that an up-to-date treatment like that under discussion is a valuable addition to technical literature. As mentioned above, however, a more extended discussion of the practical applications of the subject would have added materially to the interest and usefulness of the book.

S. E. Slocum.

The Elements of Surveying and Geodesy. By W. C. POPPLEWELL. New York, Longmans, Green and Company, 1915. 244 pp. Price $2.25.

THIS book contains the usual descriptions of surveying instruments and their uses, chapters on geodetic astronomy, earthwork calculations, railway curves, underground and hydrographic surveying. Distinctively an English text, it contains no matter pertaining to public land surveys in the United States. The exposition is clear, detailed, and accurate. The mathematical nature of the subject is kept in the foreground. A brief course, as the title indicates, it covers the field in a surprising manner.

The book contains five pages of four-place tables. There are no collections of trigonometric or other formulas. For field work it would need to be supplemented by a separate book of tables. There are no exercises.

W. V. LOVITT.

NOTES.

THE third annual meeting of the Mathematical Association of America was held at the University of Chicago on Thursday and Friday, December 27–28, 1917. At the opening session on Thursday morning papers were presented by A. F. FRUMVELLER: "The graph of $f(x)$ in line coordinates for complex numbers;" F. H. HODGE: "On the generalization of the witch and the cissoid;" W. H. BUSSEY: "Fermat's method of infinite descent;" C. N. MOORE: "On the disciplinary and applied values of mathematical study;" E. J. MOULTON: "On the content of a second course in the calculus." Thursday afternoon was devoted to a consideration of the teaching of descriptive geometry as a college subject, opening with an address by W. H. ROEVER, followed by extended discussion. At the meeting of the Council important matters of policy and procedure were taken up. The evening was dedicated to a joint dinner with the Chicago Section of the American Mathematical Society. On Friday morning reports were presented by the committees on mathematical requirements, libraries, and mathematical dictionary. General discussion followed each report. In the afternoon a joint session

was held with the Chicago Section, after which the annual business meeting of the Association and the election of officers took place.

At the meeting of the National academy of sciences, at Philadelphia, November 20–21, papers were read by Edward Kasner: "Geometric aspects of the theory of heat," and O. E. Glenn: "Invariants which are functions of parameters of the transformation."

At the Pittsburgh meeting of the American association for the advancement of science, the retiring vice-presidential address before Section A was by Professor L. P. Eisenhart on "The kinematical generation of surfaces."

The regular summer meeting of the Swiss mathematical society was held at Zurich on September 11, 1917. The programme included papers by David Hilbert: "Axiomatic thought;" C. Caratheodory: "Concerning the geometric treatment of extrema in double integrals;" and Arnold Emch: "Concerning plane curves with certain properties."

At the meeting of the London mathematical society on December 5 the following papers were read: by R. L. Hippisley: "A new method of describing a three bar curve;" by Oscar Hoppe: "Proof of the primality of $n = (10^{19} - 1)/9$;" by G. H. Hardy and E. J. Littlewood: "New Tauberian theorems;" by C. V. H. Rao: "The curves which lie on the quartic surface in space of four dimensions and the corresponding curves on the cubic surface and the quartic with a double conic;" by W. H. Young: "The connection between Legendre series and Fourier series," and "Series of Bessel functions."

At the meeting of the Edinburgh mathematical society on December 14 papers were read by C. Tweedie: "Nicole's contribution to the foundation of the calculus of finite differences;" by E. T. Whittaker: "An asymptotic representation of the exponential function."

The following American doctorates with mathematics as a major subject should be added to the list published in the December, 1917, Bulletin: J. D. Barter, California, "A

contribution to the theory of vector functions;" SISTER M. GERVAS, Catholic University of America, "On the cardioids fulfilling assigned conditions;" W. G. HUBERT, New York University, "Sextic curves with two triple points;" J. N. RICE, Catholic University of America, "On the inscribed and circumscribed triangles of the plane rational quartic curve."

THE Paris academy of sciences has awarded the Bordin prize of 3,000 francs to M. GASTON JULIA; the Francoeur prize of 1,000 francs to M. HENRI VILLAT for his publications on hydrodynamics; the Montyon prize of 700 francs to M. RENÉ DE SAUSSURE for his work in mechanics; and the Poncelet prize of 200 francs to M. JULES ANDRADE for work in applied mechanics.

PROFESSOR C. A. EPPERSON, of the Kirksville, Mo., Normal School has been commissioned first lieutenant, coast artillery reserve corps, Battery E, 61st Artillery.

PROFESSOR WILLIAM MARSHALL, of Purdue University, has entered the national food administration service and has been assigned to duty as controller and chief statistician of the international sugar committee.

PROFESSOR J. N. VAN DER VRIES, of the University of Kansas, has received a year's leave of absence to engage in the work of the field division of the U. S. chamber of commerce.

REV. A. S. HAWKESWORTH has been appointed mathematician on the staff of the chief of ordnance of the Navy Department.

AT Syracuse University associate professor F. F. DECKER has been promoted to a full professorship of mathematics. Dr. J. L. JONES has been promoted from an instructorship to an assistant professorship of mathematics.

BOOK CATALOGUES:—Galloway and Porter, Cambridge, England, catalogue 89, 50 titles in mathematics and physics.—Francis Edwards, 83 High Street, London, catalogue 377, 18 titles in mathematics.—W. Heffer and Sons, Cambridge, England, catalogue 169, 543 titles in mathematics, physics, and engineering.

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

DI DIA (G.). L'omografia e l'involuzione nelle forme fondamentali di 1a specie e la dualità nella geometria proiettiva. Bologna, tip. Cuppini, 1917. L. 1.05

GINN AND COMPANY. Mathematical portraits and pages. Boston, Ginn, 1916. 19 pp.

HENDERSON (A.). The twenty-seven lines upon the cubic surface. 2d edition. (Diss.) Chicago, 1915. 8vo. 6+100 pp. 12 plates in pocket.

PÉREZ (A.). Disquisiciones trigonométricas. Buenos Aires, Imprenta Coni Hnos, 1915.

——. Investigaciones matemáticas curiosas o útiles pero poco conocidas. Buenos Aires, 1916.

PICARD (E.). Les sciences mathématiques en France depuis un demi-siècle. Paris, Gauthier-Villars, 1917. 8vo. 26 pp. Fr. 2.00

REY PASTOR (J.). Elementos de análisis algebraico. Madrid, 1917.

WARGNY (C.). Reglas fundamentales de differenciación. Valparaiso, Imprenta "Barcelona," 1916. 176 pp.

II. ELEMENTARY MATHEMATICS.

ARNOLD (E. E.). See DURELL (F.).

DIXIT-DOMINUS (G.). Teorica grafica dimostrativa sulla trisezione degli angoli piani. Palermo, tip. Fratelli Vena, 1917. 16mo. 16 pp. con 6 tavole.

DURELL (F.) and ARNOLD (E. E.). Plane and solid geometry. New York, Merrill, 1917. 8vo. 503 pp.

——. Solid geometry. New York, Merrill, 1917. 8vo. Pp. 18+311–499.

FAZZARI (G.). Elementi di aritmetica con note storiche e numerose questione varie per le scuole medie superiori. Parte 1a: Numeri interi, operazioni, divisibilità, numeri primi. 4a edizione. Palermo, Trimarchi, 1918. 132 pp. L. 1.70

FIDENCIO DE ALZÁA (—.). Nociones de geometria plana. 1a parte, con arreglo a las ultimas programas de los colegios nacionales y escuelas normales de la República Argentina. Buenos Aires, Libreria de A. Garcia Santos, 1917.

GARZÓN (J. M.). Tratado de trigonometria elemental. 2a edición. Buenos Aires, 1916. 268 pp.

HARRIS (A. v. S.) and WALDO (L. McL.). Number games for primary grades. Chicago, Beckley-Cardy Company, 1917. 123 pp. $0.60

MILLER (A. A.). Circleometry. Des Moines, The Homestead Company, 1916. 16 pp. $0.60

THOMSON (J. B.). The art of teaching arithmetic. A book for class teachers. New York, Longmans, 1917. 8vo. 8+295 pp. $1.35

WALDO (L. McL.). See HARRIS (A. v. S.).

WHITED (K.). Trisection of an angle. The problem solved geometrically with other problems. Redmond, Ore., K. Whited, 1917. Royal 8vo. 14 pp. Paper. $1.00

III. APPLIED MATHEMATICS.

BIANCHI (E.). Tavole astronomiche di latitudine e longitudine (Ministero della guerra: direzione generale d'aeronautica, istituto centrale aeronautico). Roma, tip. R. Accademia dei Lincei, 1917. 8vo. 16+24 pp.

BRADLEY (H. C.). See KENISON (E.).

CANDIANI (G.). Gli elementi per lo studio della resistenza al moto dei solidi nei fluidi: comunicazione letta al IX congresso della Società italiana per il progresso delle scienze, Milano, 4 aprile, 1917. Padova, Società coop. tipografica, 1917. 16mo. 15 pp.

CLAVARINO (A.). Tavole per il calcolo degli elementi riguardanti una traiettoria qualunque. Vicenza, tip. G. Rumor, 1917. 24mo. 35 pp.

CREPALDI (E. L.). Lezioni di meccanica dei motori a scoppio, svolte ai corsi allievi ufficiali automobilisti e volontari automobilisti in Padova. 3a edizione. Padova, La Litotipo, 1917. 8vo. 287 pp. L. 5.50

DUFOUR (P. T.). Nouveau procédé permettant d'obtenir les perspectives-reliefs des formes géographiques représentées sur les cartes hypsométriques. Paris, Delagrave, 1917. 4to.

EFFEMERIDI astronomiche per il 1917 (Ministero della guerra: direzione generale d'aeronautica, istituto centrale aeronautico). Roma, tip. R. Accademia dei Lincei, 1917. 8vo. 41 pp.

ELEMENTI geodetici dei punti contenuti nei fogli 130 e 131 (Orvieto, Foligno) della carta d'Italia (Istituto geografico militare). Firenze, tip. Barbèra, di Alfani e Venturi, 1917. 4to. 70+85 pp. con 2 tavole.

GOFFI (C.). Tecnica moderna degli acciai: manuale per gli operai aggiustatori meccanici. (Manuali Hoepli.) Milano, Hoepli, 1917. 24mo. 19+260 pp. con tavole. L. 4.50

GRASSO (C.). Elementi di tecnologia, comprendente la materia dei programmi ministeriali per la 2a e 3a classe industriale, sezione meccanici elettricisti. 2 volumi. Napoli, casa ed. Elpis (S. Morano), 1917. 16mo. 170+205 pp. L. 5.75

HAMMICK (D. L.). See PERRIN (J.).

HEATH (T. E.). The distances, absolute magnitudes, and spectra of 734 stars. Arranged for use with ordinary star maps. Tenby, Miss Crealock, 1917. 4 + 52 pp. 2s.

KENISON (E.) and BRADLEY (H. C.). Descriptive geometry. New York, Macmillan, 1917. 10+287 pp. Cloth. $2.00

LEPPER (G. H.). From nebula to nebula or the dynamics of the heavens. Third edition rewritten and enlarged. Pittsburgh, G. H. Lepper, 1917. 363 pp. $3.50

MAGGI (G. A.). Elementi di statica e teoria dei vettori applicati. Pisa, E. Spoerri (s. tip.), 1917. 8vo. 4+175 pp.

MASSERO (F.). Manuale pratico per l'aggiustatore meccanico. (Manuali Hoepli.) Milano, Hoepli, 1917. 24mo. 12+263 pp. L. 4.50

PERRIN (J.). Atoms. Translated by D. L. Hammick. New York, Van Nostrand, 1916. 16+211 pp. $2.50

ZALDARI (P.). Annuities and amortization tables. New York, Bankers Encyclopedia Company, 1917. 350 pp. Morocco. $10.00